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Relativistic superfluid vortices and Helmholz's theorem

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Abstract. The dynamics of relativistic quantum vortices were recently analysed in a model based on the nonlinear wave equation for a complex scalar field. These results are presented here in the context of relativistic pure superfluids and the existence of the correct non-relativistic limit is verified. Relativistic superfluid vortices are essentially different from their Newtonian limit—their equation of motion contains an acceleration term, absent in non-relativistic vortex dynamics. Still, it is shown that under certain conditions, the relativistic Euler equation and Helmholz's theorem are obtained as limiting cases.

1. Introduction

It is well known from the classical theory of vortices in fluids that the dynamics of vortices are determined by the Helmholz theorem [1, 2]. This theorem, which in turn follows from the Euler equation for the fluid in which the vortices appear, states that the velocity of a vortex is equal to the local velocity of the fluid at the vortex. So far, it is generally accepted that Helmholz's theorem applies to all types of vortices. In this statement, isolated as well as non-isolated vortices are included, in spite of the singularities that characterize the velocity and vorticity fields in the first case. Since the Helmholz theorem emerged as part of Newtonian fluid mechanics, it is natural to question its validity for vortices in relativistic fluids, especially since the transformation laws for the velocities in relativistic dynamics are quite different from those in Newtonian dynamics.

Relativistic vortices have rarely been studied in the literature. Lund and Regge [3] introduced the relation between relativistic superfluid (isolated) vortices and the Kalb–Ramond interstring interaction [4]. Their result, however, ignored the fact that the relativistic (four-vector) velocity field of the fluid must be time-like and the Kalb–Ramond or Biot–Savart-like expression for the velocity field should, thus, be modified [5, 6]. Several years later, Rothen [7] studied the vortices that appear in the core of a neutron star in a relativistic hydrodynamical model. Only in recent years did the theoretical similarity between superfluid vortices and global cosmic strings cause further interest in relativistic superfluid vortices [5, 6, 8–12].

This similarity (like the theoretical similarity between quantized magnetic flux tubes in superconductors and local cosmic strings [13]) naturally brought up the questions: what are the common features of the apparently very different types of quantum vortex phenomena and how deep can we go in identifying common features?

To answer this question, a common relativistic version of the Ginzburg–Landau–Abrikosov–Pitaevskii models was recently studied [6, 10] by the present author. Its basic

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underlying concept is that a complex (Lorentz scalar) function $\phi(x)$, describing the ground state of the quantum fluid and being defined over the whole Minkowski spacetime, contains the full details of the history and evolution of the vortex system, in that asymptotically far from the vortices, $\phi(x)$ describes the medium in which the vortices appear, while the centres of the vortices are 2D time-like manifolds on which $\phi(x)$ vanishes.

Through analysis of the $\phi(x)$-field equation near a vortex, and there making use of the analytical properties of the solution, the *exact* equations of motion of any general vortex system described by $\phi(x)$ were found [10, 11]. These equations of motion, being Lorentz covariant, are essentially different from the non-relativistic equations of motion of isolated or superfluid vortices [1, 14–16] in some respects: first, the relativistic equations of motion depend on the *acceleration* of the vortices, similar to particles and in distinct contrast to the case of non-relativistic vortices, whose equations of motion depend on their velocity alone (this could also be anticipated from application of Lorentz symmetry to the dependence of the phenomenological equations of motion [15] on the curvature of the vortices, which contain a second-order differential operator along the spatial direction of the vortex). Second, the field $\phi(x)$, being a complex function, implies the existence of *two* scalar potentials with which vortices interact. Writing $\phi(x)$ as

$$\phi(x) = |\phi(x)|e^{i\varphi(x)} \qquad |\phi(x)| = \phi_0 e^{-\psi(x)} \tag{1.1}$$

these two potentials are represented by the generalized velocity potential or phase function $\varphi(x)$ and another field $\psi(x)$.

While the role of the phase function, or the generalized velocity potential $\varphi(x)$, is well known in vortex dynamics, the assignment of an independent dynamical role to the potential $\psi(x)$ [10] is new. Since it is essentially different from zero only in the region of the vortex core where it leads to a very short-ranged attractive force, this role has so far been ignored. However, $\psi(x)$ diverges on the vortices' core ($\psi(x) \to \infty$) and it is $\psi(x)$ that carries the nonlinear aspects of vortex dynamics (this holds for relativistic as well as non-relativistic vortices). In fact, the nonlinearity of vortex dynamics, manifested through the potential $\psi(x)$, has far reaching consequences. First, although the two fields ($\varphi(x)$ and $\psi(x)$) lead, separately, to diverging self-interactions of vortices, these divergences *always* cancel each other, leaving an automatically regularized self-interaction in the equation of motion [10, 12]. The nonlinearity also implies a relation between boundary conditions (the asymptotic behaviour of the medium) and the dynamics of the vortex sources, where the regularization of the self-interaction is fully determined by the theory: the cut-off parameter used to regularize self-interactions is not arbitrary, or *ad hoc* [15, 17], but explicitly given in terms of parameters of the model [12]. Also, since the incompressibility assumption is inacceptable for relativistic vortices [6], it implies that $\psi(x)$, varying with the density, cannot be ignored in a relativistic model (recent work on propagation of small disturbances in a relativistic superfluid shows that $\psi(x)$ carries the massive modes, as in spontaneous symmetry breaking, a purely relativistic effect [18]). Due to its importance in the region of the vortices' core, the field $\psi(x)$ is expected to play an important role in the theoretical explanation of the intercommuting interchange of vortices.

Since the model is non-dissipative, a complete action integral, which is valid near the vortices' cores as well, was found [10, 11] from which the vortices' equations of motion, as well as the $\varphi(x)$ and $\psi(x)$ field equations (with the vortices as singular sources), can all be derived for arbitrary vortex configurations. This action provides a unified model in which different types of quantum vortices are described. The differences between different types of vortices (namely, superfluid vortices, cosmic strings, etc) are wholly contained in a set of parameters pertaining to the medium in which the vortices appear, not to the vortex phenomena. The bare string tension (the analogue of a particle's mass) is identified and determined for all types of vortices, as well as the coupling constant of the vortices to the field corresponding to the generalized velocity potential. It is found that the ratio (bare string tension)/(vortex coupling constant) is always equal to ± 1 (the winding direction of the vortex) irrespective of the other parameters of the vortex. These results are interpreted as evidence for the actual existence, though directly unseen, of elementary vortices as the building blocks of all types of quantum vortices [10].

This model has been presented so far fully within the context of relativistic field theory. A necessary consistency check for any relativistic theory is that it yields the correct non-relativistic limit. Since cosmic strings appear only with respect to a Lorentz invariant vacuum, the non-relativistic limit can be checked only in the case of vortices in a material medium, namely, superfluid vortices. Since critical velocities in HeII are much lower than the velocity of light, we do not expect to see relativistic experimental laboratory effects on Earth, but the relativistic analysis may well be valid for vortices in neutron stars. The present paper has, thus, two main objectives: first, to present the general model in terms corresponding to (relativistic) superfluids; and second, to study the conditions under which the relativistic equation of motion of the vortices reduces to the familiar Euler equation and the statement of the Helmholz theorem. In the following, we therefore relate (section 2) our field model with the hydrodynamic description of a pure superfluid at $T = 0$ K and, in sections 3 and 4, two results are shown: (i) under certain conditions the relativistic Euler equation is obtained; and (ii) the usual equation of motion of non-relativistic vortices is obtained in the limit $c \to \infty$. Thus, the relation with the well known non-relativistic vortex dynamics is established.

2. Dynamics of relativistic superfluids with vortices

For the description of relativistic superfluid vortices, the medium in which the vortices appear is described by a complex scalar field $\phi(x)$ in a Minkowski spacetime $\mathcal{M}$. In terms of the superfluid, and in analogy with the non-relativistic models [19], this field is expected to describe the superfluid phase in the $T = 0$ limit without normal excitations. The metric in $\mathcal{M}$ is $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and the scalar product of two vectors is given by $a_\mu b^\mu = a \cdot b$.

In our model [6, 10], the field $\phi(x)$ satisfies the generic field equation

$$\nabla^2 \phi = \mathcal{U}'(|\phi|^2)\phi \tag{2.1}$$

where $\mathcal{U}(|\phi|^2)$ is a potential function that depends on the structure of the fluid and is related (see equation (2.10) below) to its equation of state. In the absence of vortices, or asymptotically away from them (assuming no net vorticity at infinity), and in the absence of any other disturbance, the field must have the very simple form

$$\phi(x) = \phi_0 \mathrm{e}^{\mathrm{i}V \cdot x} \tag{2.2}$$

with a constant vector V_μ.

The meaning of this vector is obtained from the dynamical quantities and relations that characterize the ϕ-field. Writing the field as

$$\phi(x) = |\phi(x)| \mathrm{e}^{\mathrm{i}\varphi(x)}$$

the dynamics of the field imply two conserved quantities: the number-of-particles current

$$n^\mu(x) = 2|\phi(x)|^2\varphi^{,\mu}(x) \tag{2.3}$$

and the energy–momentum tensor

$$\begin{aligned}T^{\mu\nu}(x) = {} & 2|\phi(x)|^{,\mu}|\phi(x)|^{,\nu} + 2|\phi(x)|^2\varphi(x)^{,\mu}\varphi(x)^{,\nu} \\ & - \left[|\phi(x)|^{,\lambda}|\phi(x)|_{,\lambda} + |\phi(x)|^2\varphi^{,\lambda}(x)\varphi_{,\lambda}(x) + \mathcal{U}(|\phi|^2)\right]g^{\mu\nu}.\end{aligned} \tag{2.4}$$

Substituting the asymptotic field (2.2) into the field equation (2.1) yields the condition

$$V_\mu V^\mu + \mathcal{U}'({\phi_0}^2) = 0 \tag{2.5}$$

and equations (2.3) and (2.4) yield the expressions

$$\begin{aligned}n^\mu &= 2{\phi_0}^2 V^\mu \\ T^{\mu\nu} &= 2{\phi_0}^2 V^\mu V^\nu - \left[{\phi_0}^2 V^\lambda V_\lambda + \mathcal{U}({\phi_0}^2)\right] g^{\mu\nu}.\end{aligned} \tag{2.6}$$

Comparing these expression with the standard form of the particles' current and the energy–momentum tensor of an ideal fluid in equilibrium [2]

$$\begin{aligned}n^\mu &= n \cdot U^\mu \\ T^{\mu\nu} &= (\epsilon + p)U^\mu U^\nu + p \cdot g^{\mu\nu}\end{aligned} \tag{2.7}$$

implies first that V^μ is proportional to the fluid's velocity U^μ

$$V^\mu = \mu_0 \cdot U_0^\mu \tag{2.8}$$

(the subscript zero refers to the constant values pertaining to the asymptotic homogeneous medium). The proportionality constant is determined by equation (2.5)

$${\mu_0}^2 = \mathcal{U}'({\phi_0}^2). \tag{2.9}$$

The particles' density, pressure and the energy density are then given by [20, 21]

$$\begin{aligned}n_0 &= 2{\phi_0}^2\mu_0({\phi_0}^2) \\ \epsilon_0 &= {\phi_0}^2\mathcal{U}'({\phi_0}^2) + \mathcal{U}({\phi_0}^2) \\ p_0 &= {\phi_0}^2\mathcal{U}'({\phi_0}^2) - \mathcal{U}({\phi_0}^2).\end{aligned} \tag{2.10}$$

Equation (2.10) is the equation of state of the fluid, parametrized by ${\phi_0}^2$. Since all the three state functions n, ϵ and p depend on only one parameter, this model necessarily refers to a $T = 0$ K situation.

From equations (2.9) and (2.10), it is easy to show the existence of the standard thermodynamic relations at zero temperature (referring to ϕ_0 as a variable)

$$\begin{aligned}\mathrm{d}\epsilon_0 &= \mu_0\,\mathrm{d}n_0 \\ \epsilon_0 + p_0 &= \mu_0 n_0.\end{aligned} \tag{2.11}$$

These relations verify that μ_0 is the (relativistic) chemical potential of the fluid [22].

Let us now introduce vortices into the superfluid and see how the foregoing pure hydrodynamical picture is affected. In the presence of vortices, the field (2.2) characterizes the background medium with respect to which the vortices appear. It may, therefore, be considered as the asymptotic form of $\phi(x)$, away from the vortices, while in the vortex region it should be multiplied by a proper vortex part to give $\phi(x)$

$$\phi(x) = \phi_0 \mathrm{e}^{\mathrm{i}V\cdot x}\Psi(x). \tag{2.12}$$

The vortex part $\Psi(x)$ vanishes on the vortices and $|\Psi| \to 1$, $\nabla[\arg(\Psi)] \to 0$ asymptotically. Comparing with equation (1.1), $\Psi(x)$ is given by

$$\Psi(x) = \mathrm{e}^{-\psi(x)} \cdot \mathrm{e}^{\mathrm{i}[\varphi(x) - V\cdot x]}. \tag{2.13}$$

The asymptotic conditions in a vortex system (assuming no propagation of disturbances to infinity) are therefore $\nabla_\mu \varphi(x) \to V_\mu$ and $\psi(x) \to 0$ away from the vortices and $\psi(x) \to \infty$ on the vortices' world sheets.

In the presence of vortices (or in general, whenever $|\phi(x)|$ is not constant) the particles' flow and the energy–momentum flow are no longer parallel and the particles' current (2.3) and energy–momentum tensor (2.4) cannot be brought to the equilibrium form (2.7). There is, therefore, a deviation from the pure hydrodynamical picture, just as in the non-relativistic case [23], and the superfluid cannot be regarded as being in thermal equilibrium, not even locally. Out of equilibrium, the definition of the rest frame of the fluid and the thermodynamic functions is not unique [24]. Since we consider superfluids, it is possible in the present case to use the irrotationality condition in relativistic ideal fluids [2, 25]

$$(\mu U_\nu)_{,\mu} - (\mu U_\mu)_{,\nu} = 0 \tag{2.14}$$

and make the identification

$$\mu(x) U_\mu(x) \equiv \varphi_{,\mu}(x). \tag{2.15}$$

Instead of equation (2.9), the chemical potential is now determined by the real part of equation (2.1)

$$\mu^2(x) = -[\nabla\varphi(x)]^2 = \mathcal{U}'(|\phi|^2) - \frac{1}{|\phi|}\nabla^2|\phi|. \tag{2.16}$$

The particles' current and their density are consequently determined from equations (2.3) and (2.15)

$$n^\mu(x) = n(x)U^\mu(x) \qquad n(x) = 2\mu(x)|\phi(x)|^2. \tag{2.17}$$

Since $\mu(x)$ and $n(x)$ depend not only on $|\phi|$ but also on its first and second gradients, relations (2.11) cannot be fulfilled. The determination of pressure and energy density is less clear. It is possible, as an example, to define

$$\begin{aligned} \epsilon(x) &= T_{\mu\nu}U^\mu U^\nu = \tfrac{1}{2}\mu n + \mathcal{U}(|\phi|^2) + 2(\nabla|\phi|\cdot U)^2 + (\nabla|\phi|)^2 \\ p(x) &= \tfrac{1}{3}(\epsilon + T^\mu_\mu) = \tfrac{1}{2}\mu n - \mathcal{U}(|\phi|^2) + \tfrac{2}{3}(\nabla|\phi|\cdot U)^2 - \tfrac{1}{3}(\nabla|\phi|)^2. \end{aligned} \tag{2.18}$$

Clearly, this is not the only option. However, the explicit form of the thermodynamic functions is not necessary in the following.

Consider now an arbitrary system of vortices embedded in the superfluid. A single isolated vortex is characterized by (i) the 2D time-like connected manifold (called the vortex's world sheet) $x^\mu = \xi^\mu(\zeta^a)$, $\mu = 0, 1, 2, 3$, $a = 0, 1$, on which the centre of the vortex moves in Minkowski spacetime, which satisfies the equation $\phi[x = \xi(\zeta)] = 0$; and (ii) the value of the contour-independent line integral $\oint \mathrm{d}\varphi = 2\pi\kappa$ around it, where $\varphi(x) = \arg[\phi(x)]$ is the multivalued phase function and κ is the integer-valued winding number. Also, asymptotically, away from the vortices, $|\phi(x)| \to \phi_0 =$ constant. The dynamics of such systems, when the medium described by $\phi(x)$ satisfies equation (2.1), is analysed in [10, 11] where it is shown that an arbitrary system of vortices is completely described by the following set of equations.

(i) The *exact* equation of motion of each single vortex (with winding number κ and world sheet $x^\mu = \xi^\mu(\zeta)$)

$$- \xi^{\mu|a}{}_{|a} = \frac{\kappa}{|\kappa|}\varepsilon^{\mu}{}_{\nu\lambda\rho}\varphi^{,\nu}\Sigma^{\lambda\rho} + 2\psi_{,\nu}(g^{\mu\nu} - \gamma^{ab}\xi^{\mu}_{,a}\xi^{\nu}_{,b}) \tag{2.19}$$

where $\gamma_{ab} = g_{\mu\nu}\xi^{\mu}_{,a}\xi^{\nu}_{,b}$ is the induced metric on the world sheet, $\Sigma^{\mu\nu} = \sqrt{-\gamma}\xi^{\mu}_{,a}\xi^{\nu}_{,b}\varepsilon^{ab}$ is the unit antisymmetric tensor tangent to the world sheet and $\varepsilon^{01} = -\varepsilon^{10} = 1$. The vertical bar indicates covariant differentiation on the world sheet.

The LHS of equation (2.19) is the external curvature of the vortex's world sheet in the 4D Minkowski spacetime [26] and being a second-order differential operator (on $\xi^\mu(\zeta)$), including the time-like direction, is a generalized acceleration. Thus, the equation determines the vortex's external geometry, including its motion in spacetime. The vector fields $\varphi^{,\mu}(x)$ and $\psi^{,\mu}(x)$ are computed on the world sheet. A highly important property of this equation is that although $\varphi^{,\mu}(x)$ and $\psi^{,\mu}(x)$ are, separately, singular on the vortex, they are combined in equation (2.19) in such a way that the singularities cancel and the RHS is regular on the vortex.

A particular aspect of this equation is the relativistic Magnus force. If we substitute instead of $\varphi^{,\mu}(x)$ and $\psi^{,\mu}(x)$ their asymptotic values (i.e. focus on the effect of the fluid and ignore the contributions of all the other vortices, boundaries, etc), we get

$$- \xi^{\mu|a}{}_{|a} = \frac{\kappa}{|\kappa|}\varepsilon^{\mu}{}_{\nu\lambda\rho}V^{\nu}\Sigma^{\lambda\rho}. \tag{2.20}$$

The RHS represents the interaction of the vortex with the medium in which it appears. This interaction vanishes only if V^μ is tangental to the vortex's world sheet.

It is also interesting to note that the structure of the superfluid, encompassed in the potential $\mathcal{U}(|\phi|^2)$, is not manifested at all in the vortices' equation of motion. The vortices are influenced by the details of the structure of the superfluid only through the field equation of $\psi(x)$, equation (2.22) below.

(ii) The φ-field equation (circulation condition)

$$(\varphi_{,\nu})_{,\mu} - (\varphi_{,\mu})_{,\nu} = \pi\sum_i \kappa_i\varepsilon_{\mu\nu\lambda\rho}\iint \delta^4(x - \xi_i)\,\mathrm{d}\Sigma_i^{\lambda\rho}. \tag{2.21}$$

Here, the sum is over all the vortices $\xi_i^\mu(\zeta^a)$ with winding numbers κ_i, $\mathrm{d}\Sigma_i^{\mu\nu}$ being the directed surface element on the ith vortex's world sheet. $\varepsilon_{\mu\nu\lambda\rho}$ is the fully antisymmetric unit pseudo-tensor with $\varepsilon_{0123} = 1$.

(iii) The ψ-field equation

$$\psi_{,\mu}^{;\mu} - \psi^{,\mu}\psi_{,\mu} + \mathcal{U}'(\phi_0^2 \mathrm{e}^{-2\psi}) = -\varphi^{,\mu}\varphi_{,\mu} - \sum_i 2\pi|\kappa_i| \iint \delta^4(x - \xi_i)\,\mathrm{d}\Sigma_i. \tag{2.22}$$

Comparing equations (2.21) and (2.22), the vortices are antisymmetric tensorial sources for the φ-field, but scalar sources for the ψ-field.

Together, all these equations are derivable from a common action integral. The proper canonical field variable [11], instead of $\varphi(x)$, is an antisymmetric tensor field $A_{\mu\nu}$, satisfying

$$\begin{aligned} &2\mathrm{e}^{-2\psi}\varphi^{,\mu} = \tfrac{1}{2}\varepsilon^{\mu\nu\lambda\rho}A_{\lambda\nu,\rho} = -\tfrac{1}{6}\varepsilon^{\mu\nu\lambda\rho}H_{\nu\lambda\rho} \\ &H_{\mu\nu\lambda} \equiv A_{\mu\nu,\lambda} + A_{\nu\lambda,\mu} + A_{\lambda\mu,\nu}. \end{aligned} \tag{2.23}$$

The principle that governs this relation is that the LHS is the conserved number-of-particles current, the conservation being automatically ensured by the structure of the RHS. The action that describes an arbitrary vortex system with winding numbers κ_i and world sheets $\xi_i(\zeta)$ in an ideal superfluid is then [10, 11]

$$\begin{aligned} S = 2\pi\phi_0^2 \sum_i \biggl[-|\kappa_i| \iint \mathrm{e}^{-2\psi(\xi_i)}\,\mathrm{d}\Sigma_i + \kappa_i \iint \tfrac{1}{2}A_{\mu\nu}(\xi_i)\,\mathrm{d}\Sigma_i^{\mu\nu} \biggr] \\ - \int \biggl[\frac{\phi_0^2}{24}\mathrm{e}^{2\psi}H^{\mu\nu\lambda}H_{\mu\nu\lambda} + \phi_0^2\mathrm{e}^{-2\psi}\psi^{,\mu}\psi_{,\mu} + \mathcal{U}(\phi_0^2\mathrm{e}^{-2\psi}) \biggr]\,\mathrm{d}^4x. \end{aligned} \tag{2.24}$$

In a real fluid, this action should be introduced as the vortices' contribution. In the following, we concentrate on equation (2.19) as describing the dynamics of vortices in an ideal superfluid.

3. Isolated against non-isolated vortices and the relativistic Euler equation

Before discussing the limits of equation (2.19), we pause to stress the distinction between isolated and non-isolated vortices, which is particularly explicit in the relativistic regime. Non-isolated vortices are classically defined in a medium in which the vorticity field $\boldsymbol{w}(t, \boldsymbol{x}) \equiv \boldsymbol{\nabla} \times \boldsymbol{v}(t, \boldsymbol{x})$ is regular and non-vanishing, at least in some parts of the fluid, and the Euler equation is of the form [1, 2]

$$\dot{\boldsymbol{v}}(t, \boldsymbol{x})(t, \boldsymbol{x}) + (\boldsymbol{v}(t, \boldsymbol{x}) \cdot \boldsymbol{\nabla})\boldsymbol{v}(t, \boldsymbol{x}) = -\boldsymbol{\nabla}\mu(t, \boldsymbol{x}). \tag{3.1}$$

The continuous non-isolated vortices are then defined as the integral lines of $\boldsymbol{w}(t, \boldsymbol{x})$ in the regions where $\boldsymbol{w}(t, \boldsymbol{x})$ does not vanish. As a consequence of equation (3.1), these lines move together with the fluid. This is essentially the Helmholz theorem. It also holds in a relativistic (possibly rotational) fluid, where, corresponding to equation (3.1), the equation [2, 25]

$$w_{\mu\nu}u^\nu = 0 \tag{3.2}$$

is the vorticity tensor, where u^μ is the four-velocity field and $w_{\mu\nu} = (\psi u_\nu)_{,\mu} - (\psi u_\mu)_{,\nu}$ (ψ is the relativistic chemical potential). We define [27] the dual vorticity tensor as

$$^*w^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\lambda\rho}w_{\lambda\rho}. \tag{3.3}$$

Then, at each point where ${}^*w^{\mu\nu} \neq 0$, (3.3) defines a 2D plane spanned by ${}^*w^{\mu\nu}$. Equation (3.2) then yields

$$\varepsilon_{\mu\nu\lambda\rho}{}^*w^{\mu\nu}u^{\lambda} = 0 \tag{3.4}$$

which implies that the fluid's velocity u^{μ} lies in this plane. The vortices are then defined as the integral 2D manifolds of ${}^*w^{\mu\nu}$, i.e. the 2D manifolds whose tangent space at each point coincides with the 2D plane spanned by ${}^*w^{\mu\nu}$. The fluid's velocity is then tangent at any point to the vortices which, therefore, move with the fluid. The Helmholz theorem extends, therefore, to relativistic non-isolated vortices. This result was used by Rothen [5] for vortices in the core of neutron stars.

The application of the Helmholz theorem for isolated vortices presents a completely different picture. These are defined, in multiply-connected irrotational fluids containing contours unshrinkable to a point, as the lines (closed or infinitely long) where the velocity field is not defined. Since the velocity field is not defined at the location of the vortex, and in the near vicinity of the vortex the fluid rotates around the vortex rather than moving with respect to it, the assertion of the Helmholz theorem presents a rather obscure statement for isolated vortices.

In classical vortex dynamics, it is assumed that isolated vortices can be obtained by a limiting process from non-isolated vortices. This limiting process then leads to the Biot–Savart-like formula for the fluid's velocity field, which in turn determines, via the Helmholz theorem, the dynamics of the vortices [1, 15]. Direct extension of the classical approach for relativistic vortices leads to contradictions: the relativistic Biot–Savart-like velocity field for the fluid [3] fails to comply with the requirements of relativistic dynamics because it is space-like rather than time-like [5, 6]. The correction of the incompatibility requires the introduction of another field, related to the density of the fluid [6]. The theory presented in the previous section encompasses, in a self-contained way, the concept of isolated vortices, their dynamics and interactions through fields that correspond to the fluid's velocity field and the density-related field. It is, thus, considered to properly describe the dynamics of isolated relativistic vortices. In the following, we show that under proper conditions, the relativistic Euler equation (3.4) and the non-relativistic Helmholz theorem are obtained.

To obtain the limits of equation (2.19), particular parametrizations of the world sheet need to be used. For the reduction to the relativistic Euler equation, it is convenient to describe a vortex's world sheet using a conformally flat parametrization $x^{\mu} = \xi^{\mu}(\tau, \sigma)$, satisfying the conditions

$$\frac{1}{c^2} u \cdot u = -v \cdot v = -\mathrm{e}^{2\lambda(\tau,\sigma)} \qquad u \cdot v = 0 \tag{3.5}$$

where at each point of the world sheet the tangent plane is spanned by $u^{\mu} = \dot{\xi}^{\mu}$ and $v^{\mu} = \xi^{\mu\prime}$ (a dot denotes differentiation with respect to the time-like parameter τ and the prime denotes differentiation with respect to σ). $\lambda(\tau, \sigma)$ is a function that depends on the intrinsic curvature of the vortex world sheet. In this parametrization, the equation of motion (2.19) is

$$\frac{1}{c^2}\dot{u}_{\mu} - v'_{\mu} = \frac{2\kappa}{|\kappa|}\varepsilon_{\mu\nu\lambda\rho}\varphi^{,\nu}u^{\lambda}v^{\rho} + 2\psi_{,\nu}\left[\mathrm{e}^{2\lambda(\tau,\sigma)}\delta^{\nu}_{\mu} + \frac{1}{c^2}u_{\mu}u^{\nu} - v_{\mu}v^{\nu}\right]. \tag{3.6}$$

Assume now that the curvature of the vortex can be ignored so that the term v'_{μ} may be dropped. Also, assume that the vortices are far enough so that their contribution to ψ is small and can be dropped as well. The only contribution of ψ comes from regularizing the singular part in $\varphi^{,\mu}$, denoting the regularized vector as $[\varphi]^{\mu}$. Finally, for a non-relativistic motion, the acceleration term, being divided by c^2, can also be ignored. What is left is the condition

$$\varepsilon_{\mu\nu\lambda\rho}[\varphi]^{\nu}u^{\lambda}v^{\rho} = 0. \tag{3.7}$$

Following the discussion in section 2, the velocity field of the fluid cannot be uniquely defined near the vortex and the definition becomes more ambiguous as one approaches the vortex. Since, asymptotically, $\varphi^{,\mu}$ is proportional to the velocity of the background fluid, we can consider the vector $[\varphi]^{\mu}$ as being proportional to the average fluid's velocity field in the neighbourhood of the vortex. The vectors u^{μ} and v^{μ} span the tangent plane to the world sheet. Thus, equation (3.7) is of the form of the relativistic Euler equation (3.4) with the same content: stating that the fluid's regularized four-velocity must be tangent to the vortex's world sheet. This is exactly the Helmholz theorem. In this form, equation (3.7) suits a situation in which the vortices move slowly, but the medium requires an otherwise relativistic description, such as in strong gravitational fields.

4. Equations of motion in the non-relativistic limit

If the medium is also Newtonian, the full limit of equation (2.19) for $c \to \infty$ can be taken. In this case, it is convenient to use a different parametrization, which follows from the fact that any 2D world sheet can be represented in any Lorentz frame in a parametrization $x^{\mu} = \xi^{\mu} = (t, \boldsymbol{\xi}(t,\sigma))$ satisfying $\dot{\boldsymbol{\xi}} \cdot \boldsymbol{\xi}_{,\sigma} = 0$. In this parametrization, the metric on the world sheet is

$$\mathrm{d}s^2 = -c^2A^2(t,\sigma)\,\mathrm{d}t^2 + B^2(t,\sigma)\,\mathrm{d}\sigma^2 \qquad A^2(t,\sigma) = 1 - \frac{1}{c^2}|\dot{\boldsymbol{\xi}}|^2 \qquad B^2(t,\sigma) = |\boldsymbol{\xi}_{,\sigma}|^2 \tag{4.1}$$

so that the generalized acceleration is

$$-\xi^{\mu|a}{}_{|a} = \frac{1}{AB}\left[\left(\frac{B}{c^2A}\xi^{\mu}{}_{,t}\right)_{,t} - \left(\frac{A}{B}\xi^{\mu}{}_{,\sigma}\right)_{,\sigma}\right] = -|\boldsymbol{\xi}_{,\sigma}|^{-1}(|\boldsymbol{\xi}_{,\sigma}|^{-1}\boldsymbol{\xi}^{\mu}{}_{,\sigma})_{,\sigma} + \mathrm{O}(c^{-2}). \tag{4.2}$$

We now substitute (4.2) in the spatial components of equation (2.19). Assuming the motion of the vortices to be slow, all the time derivatives which are divided by c^2 can be ignored in the limit $c \to \infty$. The remaining expression is

$$|\boldsymbol{\xi}_{,\sigma}|^{-1}(|\boldsymbol{\xi}_{,\sigma}|^{-1}\boldsymbol{\xi}^{\mu}_{,\sigma})_{,\sigma} + \frac{2\kappa}{|\kappa|}|\boldsymbol{\xi}_{,\sigma}|^{-1}\left(\nabla\varphi - \frac{m}{\hbar}\dot{\boldsymbol{\xi}}\right) \times \boldsymbol{\xi}_{,\sigma} + 2[\nabla\psi - |\boldsymbol{\xi}_{,\sigma}|^{-2}(\boldsymbol{\xi}_{,\sigma}\cdot\nabla\psi]\boldsymbol{\xi}_{,\sigma}) = 0 \tag{4.3}$$

from which, by re-ordering, we get

$$\dot{\boldsymbol{\xi}} = \boldsymbol{n} \times \left[\nabla\left(\frac{\hbar\varphi}{m}\right) \times \boldsymbol{n} + \frac{\hbar\kappa}{m|\kappa|}\nabla\psi + \frac{\hbar\kappa}{2m|\kappa|}|\boldsymbol{\xi}_{,\sigma}|^{-1}\boldsymbol{n}_{,\sigma}\right] \qquad \boldsymbol{n} \equiv |\boldsymbol{\xi}_{,\sigma}|^{-1}\boldsymbol{\xi}_{,\sigma}. \tag{4.4}$$

Unless the vortices are close to each other, the role of the $\nabla\psi$ term is merely to regularize the self-interaction included in the first term. The regularized self-interaction is proportional to the curvature [12], yielding

$$\left[\nabla\varphi - \frac{\kappa}{|\kappa|}\nabla\psi \times \boldsymbol{n}\right]_{\rm reg} \times \boldsymbol{n} = \nabla[\varphi] \times \boldsymbol{n} + \tfrac{1}{2}\alpha\kappa|\xi_{,\sigma}|^{-1}\boldsymbol{n}_{,\sigma}. \tag{4.5}$$

Here, $[\varphi]$ is the contribution to the phase function from all the other vortices and the homogeneous background superfluid and α is a positive dimensionless coefficient that depends on the structure of the vortex. Substituting (4.5) back into equation (4.4) finally gives

$$\dot{\xi} = \boldsymbol{n} \times \left[\nabla\left(\frac{\hbar[\varphi]}{m}\right) \times \boldsymbol{n} + \frac{\hbar\kappa}{2m|\kappa|}(1 + |\kappa|\alpha)|\xi_{,\sigma}|^{-1}\boldsymbol{n}_{,\sigma}\right] \tag{4.6}$$

which is the conventional fundamental formula for the velocity of non-relativistic vortices [14–16], composed of the velocity due to all the rest of the system plus a term and derived from the regularization of the self-interaction, which can be combined with the Iordanskii force [16]. Thus, the correct non-relativistic limit of our model equation (2.19) has been verified.

The automatic self-regularization coefficient α is given, approximately, by [12]

$$\alpha \approx \ln\left(\frac{R_{\rm c}}{\epsilon}\right) \tag{4.7}$$

where $R_{\rm c}$ is a typical radius of curvature of the vortex and ϵ is a cut-off parameter. So far, in regularizing the self-interaction of the vortices, the cut-off parameter ϵ has had to be introduced by hand [15, 17]. Here, however, due to the self-regularization by $\nabla\psi$ in (4.5), the theory provides [12] the exact relation between ϵ and the so-called 'healing length' a [16] derived from the theory

$$\ln\frac{\epsilon}{a} = 0.5 - \ln 2 + \frac{|\kappa| - \varepsilon_\kappa}{\kappa^2} \tag{4.8}$$

with quantities ε_κ defined by

$$\varepsilon_\kappa \equiv \lim_{r\to\infty}\left\{\int_0^r \left[(f_\kappa')^2 + \frac{\kappa^2}{x^2}{f_k}^2\right] x\,{\rm d}x - \kappa^2 \ln r\right\} \tag{4.9}$$

where the functions $f_\kappa(x)$ are the field-amplitude profile for cylindrically symmetric vortices with field

$$\phi = \phi_0 {\rm e}^{-{\rm i}\mu_0 t} f_k(r){\rm e}^{{\rm i}\kappa\theta} \tag{4.10}$$

satisfying the equation

$$\frac{1}{x}\frac{\rm d}{{\rm d}x}\left(x\frac{{\rm d}f_\kappa}{{\rm d}x}\right) - \frac{\kappa^2}{x^2}f_\kappa = [\mathcal{U}'({\phi_0}^2 f_\kappa) - {\mu_0}^2]f_\kappa. \tag{4.11}$$

In particular, for large vortex rings with $|\kappa| = 1$, one obtains [12] $\epsilon \approx 2.5a$. This result explains the similar ratios found between the measured core radii and healing lengths [16].

5. Discussion

Relativistic vortex dynamics are distinguished by the appearance of an acceleration term in the equation of motion (equation (2.19)) in a very similar way to the equations of motion for point particles. This term is entirely a relativistic effect, as is evident from the fact that it vanishes in the limit $c \to \infty$. Its appearance, however, is necessary to ensure Lorentz covariance of vortex dynamics if the equation of motion is to include a spatial curvature term [15], because the acceleration and the spatial curvature together form a Lorentz covariant quantity.

The relativistic dynamics of isolated quantum vortices was found here to reduce to the Eulerian picture of continuous vorticity when the vortices are far enough from each other so that their curvature (including acceleration) can be ignored. Classical vortex dynamics, including Helmholz's theorem, are recovered when the vortices are far enough from each other and their curvatures are not too large, so that their motion is non-relativistic in the sense that time derivatives which are of order $O(c^{-2})$, including the acceleration term, can be ignored.

The equation of motion (2.19) was derived in [10, 11] from a relativistic field theory, which describes the medium in which the vortices appear, that is based essentially on a nonlinear wave equation in a complex scalar field. Application of the same method to vortex systems described via the nonlinear Schrödinger equation yields the corresponding non-relativistic limit equation (4.3). The difference between this equation and other vortex dynamics that were obtained from the nonlinear Schrödinger equation [28] is that this equation (as is equation (2.19)) is exact within the context of its field model, while all the other approaches introduce approximations to obtain the equation of motion. These approximations are unnecessary, as an exact equation is available. They also eliminate the possibility of seeing features of the exact dynamics, especially those that are related to the existence of the ψ-field, such as the automatic self-regularization of the self-interaction (this argument applies also to Neu's approach to the nonlinear wave equation [29]).

The present model concerns vortices in an ideal superfluid where the medium is otherwise inert and homogeneous. More realistic relativistic superfluids (without vortices) have been discussed in the literature in the last decade [30]. The combination of these models is expected to provide a more realistic model of relativistic superfluids with vortices.

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